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Quaternion analysis for generalized electromagnetic fields of dyons in an isotropic medium

Jivan Singh[1], **P S Bisht**[2] **and O P S Negi**[2]

[1] Department of Physics, Government P G College, Pithoragarh (UA), India
[2] Department of Physics, Kumaun University, S S J Campus, Almora-263601 (UA), India

E-mail: jgaria@indiatimes.com, ps_bisht123@rediffmail.com and ops_negi@yahoo.co.in

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Abstract
Quaternion analysis of time-dependent Maxwell's equations in the presence of electric and magnetic charges has been developed and the solutions for the classical problem of moving charges (electric and magnetic) are obtained in a unique, simple and consistent manner.

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1. Introduction

It is believed that in spite of the recent potential importance of magnetic monopoles [1–3] and dyons [4] towards the quark confinement problem [5] of quantum choromodynamics, possible magnetic condensation of vacuum [6], CP-violation [7], their role as catalysts in proton decay [8] and the current grand unified theories [9], the formalism necessary to describe them has been clumsy and manifestly non-covariant. Keeping in view the potential importance of monopoles and the results of Witten [7] that monopoles are necessarily dyons, we [10, 11] have also constructed a self-consistent co-variant theory of generalized electromagnetic fields associated with dyons each carrying the generalized charge as a complex quantity with its real and imaginary parts as electric and magnetic constituents. On the other hand quaternions were invented by Hamilton [12] to extend the theory of complex numbers to three dimensions. The quaternionic formulation of electrodynamics has a long history [13–18], stretching back to Maxwell himself who used quaternion in his original manuscript on the application of quaternion to electromagnetism. Quaternion analysis has since been rediscovered at regular intervals and accordingly Maxwell's equations of electromagnetism are rewritten as one-quaternion equations [16–20]. Finkelstein *et al* [21] developed the quaternionic quantum mechanics and Adler [22] described the theory of the algebraic structure of quantum choromodynamics for strong interactions. Various aspects of quaternions are discussed by Morita [23] towards the kinematical structure of Poincare gauge theory and the left-right Weinberg–Salam theory of quantum choromodynamics. We have also studied

[24] the quaternionic formulation for generalized field equations of dyons in unique, simpler and compact notations. Quaternion non-Abelian gauge theory has also been consistently discussed [25] to maintain the structural symmetry between the theory of linear gravity and electromagnetism. It is also shown that quaternion formalism characterizes the Abelian and non-Abelian gauge structures [26] of dyons in terms of the real and imaginary constituents of quaternion basis elements. Alternatively, Kravchenko [27] and his co-workers have consistently analysed the Maxwell's equations for time-dependent electromagnetic fields in a homogeneous (isotropic) and chiral medium. Extending this, we [28] have also derived the generalized Maxwell's Dirac equation in the homogenous (isotropic) medium. It has been shown that the field equations of dyons also remains invariant under the duality transformations in an isotropic homogeneous medium and the equation of motion reproduces the rotationally symmetric gauge invariant angular momentum of dyons. In order to extend the theory of monopoles and dyons in an isotropic medium and consequently the relevance of quaternion formalism of dyons, in the present paper we have undertaken a study of the quaternion analysis of time-dependent Maxwell's equations in the presence of electric and magnetic charges and the solutions for the classical problem of moving sources are obtained in a unique, simpler and consistent manner. Quaternion forms of potential current, field equation and equation of motion are developed in a compact manner and it is emphasized that the quantum equations in terms of quaternions are invariant under quaternion, Lorentz and duality transformations. It has also been emphasized that the quaternion analyticity of dyons in an isotropic medium reproduces the results of Kravchenko [27] in the absence of magnetic monopoles and accordingly this theory can be described symmetrically for pure monopoles in the absence of electric charge or vice versa.

2. Generalized field equation of dyons in a homogeneous medium

Assuming the existence of magnetic monopoles, let us write the following form [29] of symmetric generalized Maxwell–Dirac differential equations [1, 28] in free space in SI units ($c = \hbar = 1$):

$$\begin{aligned}
\overrightarrow{\nabla} \cdot \overrightarrow{D} &= \rho_e \\
\overrightarrow{\nabla} \cdot \overrightarrow{B} &= \mu_0 \rho_m \\
\overrightarrow{\nabla} \times \overrightarrow{E} &= -\frac{\partial \overrightarrow{B}}{\partial t} - \frac{\overrightarrow{j_m}}{\epsilon_0} \\
\overrightarrow{\nabla} \times \overrightarrow{H} &= \frac{\partial \overrightarrow{D}}{\partial t} + \overrightarrow{j_e}
\end{aligned} \tag{1}$$

where ρ_e and ρ_m are respectively the electric and magnetic charge densities while $\overrightarrow{j_e}$ and $\overrightarrow{j_m}$ are the corresponding current densities, $\overrightarrow{D}$ is the electric induction vector, $\overrightarrow{E}$ is the electric field, $\overrightarrow{B}$ is the magnetic induction vector and $\overrightarrow{H}$ is the magnetic field. Here we assume a homogenous (isotropic) medium with the following definitions [27]:

$$\overrightarrow{D} = \epsilon \overrightarrow{E} \qquad (\epsilon = \epsilon_0 \epsilon_r) \tag{2}$$

and

$$\overrightarrow{B} = \mu \overrightarrow{H} \qquad (\mu = \mu_0 \mu_r) \tag{3}$$

where ϵ_0 the free space permittivity, μ_0 is the permeability of free space and ϵ_r and μ_r are defined respectively as relative permittivity and permeability in the electric and magnetic fields. On using equations (2) and (3), equations (1)–(4) take the following differential form [26]:

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \frac{\rho_e}{\epsilon} \\ \vec{\nabla}\cdot\vec{B} &= \mu\rho_m \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} - \frac{\vec{j_m}}{\epsilon} \\ \vec{\nabla}\times\vec{B} &= \frac{1}{v^2}\frac{\partial\vec{E}}{\partial t} + \mu\,\vec{j_e} \end{aligned} \tag{4}$$

where $v = \frac{1}{\sqrt{\mu\epsilon}}$ is the speed of propagation of electromagnetic waves in a homogeneous (isotropic) medium while in the case of vacuum it corresponds to the velocity of light, i.e., $v \to c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$. Differential equations (1)–(4) are the generalized field equations of dyons in a homogenous medium and the corresponding electric and magnetic fields are then called generalized electromagnetic fields of dyons. These electric and magnetic fields of dyons are expressed in the following differential form in a homogenous medium in terms of two potentials [11]:

$$\vec{E} = -\vec{\nabla}\phi_e - \frac{\partial\vec{C}}{\partial t} - \vec{\nabla}\times\vec{D} \tag{5}$$

$$\vec{B} = -\vec{\nabla}\phi_m - \frac{1}{v^2}\frac{\partial\vec{D}}{\partial t} + \vec{\nabla}\times\vec{C} \tag{6}$$

where $\{C^\mu\} = \{\phi_e, v\vec{C}\}$ and $\{D^\mu\} = \{v\phi_m, \vec{D}\}$ are the two four-potentials associated with electric and magnetic charges.

Let us define the complex vector field $\vec{\psi}$ in the following form:

$$\vec{\psi} = \vec{E} - \mathrm{i}v\vec{B}. \tag{7}$$

Equations (5), (6) and (7) thus give rise to the following relation between the generalized field and the components of the generalized four-potential:

$$\vec{\psi} = -\frac{\partial\vec{V}}{\partial t} - \vec{\nabla}\phi - \mathrm{i}v(\vec{\nabla}\times\vec{V}). \tag{8}$$

Here $\{V_\mu\}$ is the generalized four-potential of dyons in a homogenous medium and defined as

$$\{V_\mu\} = \{\phi, -\vec{V}\} \tag{9}$$

where

$$\phi = \phi_e - \mathrm{i}v\phi_m \tag{10}$$

and

$$\vec{V} = \vec{C} - \mathrm{i}\frac{\vec{D}}{v}. \tag{11}$$

Maxwell's field equation (7)–(10) may then be written in terms of the generalized field $\vec{\psi}$ as

$$\vec{\nabla}\cdot\vec{\psi} = \frac{\rho}{\epsilon} \tag{12}$$

$$\vec{\nabla}\times\vec{\psi} = -\mathrm{i}v\left(\mu\vec{J} + \frac{1}{v^2}\frac{\partial\vec{\psi}}{\partial t}\right) \tag{13}$$

where ρ and $\vec{J}$ are the generalized charge and current source densities of dyons in a homogenous medium given by

$$\rho = \rho_e - \mathrm{i}\frac{\rho_m}{v} \tag{14}$$

$$\vec{J} = \overrightarrow{j_e} - \mathrm{i}v\overrightarrow{j_m}. \tag{15}$$

From equation (12), we may now introduce a new parameter $\overrightarrow{S}$ (i.e. the field current) as

$$\overrightarrow{S} = \Box\overrightarrow{\psi} = -\mu\frac{\partial\overrightarrow{J}}{\partial t} - \frac{1}{\epsilon}\overrightarrow{\nabla}\rho - \mathrm{i}v\mu(\overrightarrow{\nabla}\times\vec{J}) \tag{16}$$

where $\Box$ is the D'Alembertian operator expressed as

$$\Box = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}, \tag{17}$$

and v is the speed of electromagnetic wave in a homogenous isotropic medium. As such the field equations (1) are generalized to the following forms in terms of the components of the complex potential of dyons as

$$\Box\phi = v\mu\rho \tag{18}$$

$$\Box\overrightarrow{V} = \mu\vec{J}. \tag{19}$$

Here we may write the following tensorial form of generalized Maxwell's Dirac equations of dyons in a homogenous medium as [28]

$$F_{\mu\nu,\nu} = j_\mu^e \tag{20}$$

$$F^d_{\mu\nu,\nu} = j_\mu^m \tag{21}$$

where $\left\{j_\mu^e\right\} = \{\rho_e, \overrightarrow{j_e}\}$ and $\left\{j_\mu^m\right\} = \{\rho_m, \overrightarrow{j_m}\}$.

Defining the generalized field tensor of a dyon as

$$G_{\mu\nu} = F_{\mu\nu} - \mathrm{i}vF^d_{\mu\nu}, \tag{22}$$

one can directly obtain the following generalized field equation of the dyon in a homogenous (isotropic) medium, i.e.,

$$G_{\mu\nu,\nu} = J_\mu \tag{23}$$

$$G^d_{\mu\nu,\nu} = 0, \tag{24}$$

where $\{J_\mu\} = \{\rho, \overrightarrow{-J}\}$.

The Lorentz four-force equation of motion for dyons in the homogenous (isotropic) medium is then written as

$$f_\mu = m_0\ddot{x_\mu} = \mathrm{Re}\, Q^*(G_{\mu\nu}u^\nu) \tag{25}$$

where 'Re' denotes the real part, $\ddot{x_\mu}$ is the four-acceleration and $\{u^\nu\}$ is the four-velocity of the particle and Q is the generalized charge of a dyon in an isotropic medium.

3. Quaternion analysis for generalized Maxwell's equation in a homogenous medium

A quaternion is defined [30, 31] as

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \tag{26}$$

where q_0, q_1, q_2, q_3 are real numbers and called the components of the quaternion q and the quaternion units e_0, e_1, e_2, e_3 satisfy the following multiplication rules:

$$\begin{aligned} e_0^2 &= 1 \\ e_j e_k &= -\delta_{jk} + \epsilon_{jkl} e_l \end{aligned} \tag{27}$$

where δ_{jk} and ϵ_{jkl} ($j, k, l = 1, 2, 3$ and $e_0 = 1$) are respectively the Kronecker delta and three-index Levi-Civita symbol. The sum of two quaternions p and q is defined by adding the corresponding components,

$$p + q = (p_0 + q_0)e_0 + (p_1 + q_1)e_1 + (p_2 + q_2)e_2 + (p_3 + q_3)e_3, \tag{28}$$

and the multiplication of two quaternions is defined as

$$p \cdot q = (p_0 q_0 - \overrightarrow{p} \cdot \overrightarrow{q})e_0 + \sum_j (p_j q_0 + p_0 q_j + \epsilon_{jkl} p_k q_l)e_j. \tag{29}$$

Thus each quaternion q is the sum of a scalar q_0 and a vector $\overrightarrow{q}$,

$$q = \text{scalar}\,(q) + \text{vector}\,(q) = q_0 + \overrightarrow{q}, \tag{30}$$

where $\overrightarrow{q} = q_1 e_1 + q_2 e_2 + q_3 e_3$. Thus the product of two quaternions p and q is also written as

$$pq = p_0 q_0 - \overrightarrow{p} \cdot \overrightarrow{q} + p_0 \overrightarrow{q} + q_0 \overrightarrow{p} + \overrightarrow{p} \times \overrightarrow{q} \tag{31}$$

where the dot and cross indicate, respectively, the usual three-dimensional scalar and vector products. For any quaternion, there exists a quaternion conjugate

$$\overline{q} = q - q_1 e_1 - q_2 e_2 - q_3 e_3 = q_0 - \overrightarrow{q}. \tag{32}$$

A quaternion conjugate is an automorphism of a ring of quaternions, i.e.,

$$(\overline{pq}) = (\overline{q})(\overline{p}). \tag{33}$$

The norm of a quaternion is given as

$$q \cdot \overline{q} = \overline{q} \cdot q = q_0^2 + q_1^2 + q_2^2 + q_3^2 = |q|^2. \tag{34}$$

The inverse of a quaternion q is also a quaternion,

$$q^{-1} = \frac{\overline{q}}{|q|^2}. \tag{35}$$

Let us define the quaternionic form of the differential operator as [26]

$$\boxdot = \left(-\frac{\mathrm{i}}{v}\partial_t + D\right) \tag{36}$$

and

$$\overline{\boxdot} = \left(-\frac{\mathrm{i}}{v}\partial_t - D\right) \tag{37}$$

where $D = e_1\partial_1 + e_2\partial_2 + e_3\partial_3$. As such we can express the quantum equation associated with generalized four-potential, four current, electric field and magnetic field in terms of quaternionic analysis as

$$V = -\mathrm{i}\frac{\phi}{v} e_0 + V_1 e_1 + V_2 e_2 + V_3 e_3 \tag{38}$$

$$J = -\mathrm{i}\rho v e_0 + J_1 e_1 + J_2 e_2 + J_3 e_3 \tag{39}$$

$$E = E_1 e_1 + E_2 e_2 + E_3 e_3 \tag{40}$$

$$B = B_1 e_1 + B_2 e_2 + B_3 e_3. \tag{41}$$

Applying operator (37) to the quaternion (7) and using equation (27), we get

$$\overline{\boxdot}\psi = \mathrm{i}v\mu J = \mathrm{i}\sqrt{\frac{\mu}{\epsilon}}J. \tag{42}$$

Similarly we may apply the operator (36) to the quaternions (38) and (39) and using the quaternion multiplication rules given by (27), we get

$$\boxdot V = \psi \qquad \boxdot J = S, \tag{43}$$

where we have used the following subsidiary condition

$$\overrightarrow{\nabla} \cdot \overrightarrow{V} + \frac{1}{v^2}\frac{\partial \phi}{\partial t} = 0 \tag{44}$$

and

$$\overrightarrow{\nabla} \cdot \overrightarrow{J} + \frac{\partial \rho}{\partial t} = 0. \tag{45}$$

Equation (44) is known as the Lorentz gauge condition while equation (45) is referred as a continuity equation. In equations (42) and (43) $\overrightarrow{\psi}$ and $\overrightarrow{S}$ are defined as quaternion valued functions in the following manner:

$$\psi = -\psi_t - \frac{\mathrm{i}}{v}(e_1\psi_1 + e_2\psi_2 + e_3\psi_3) \tag{46}$$

$$S = -S_t - \mathrm{i}\sqrt{\frac{\epsilon}{\mu}}(e_1 S_1 + e_2 S_2 + e_3 S_3) \tag{47}$$

where $\psi_t = \overrightarrow{\nabla} \cdot \overrightarrow{V} + \frac{1}{v^2}\frac{\partial\phi}{\partial t} = 0$ and $S_t = \overrightarrow{\nabla} \cdot \overrightarrow{J} + \frac{1}{v^2}\frac{\partial\rho}{\partial t} = 0$.

Similarly, we may obtain the quaternion conjugate field equations for dyons in a homogenous (isotropic) medium as

$$\overline{\boxdot V} = \overline{\psi} \tag{48}$$

$$\overline{\boxdot J} = \overline{S} \tag{49}$$

where $\overline{V}$, $\overline{J}$, $\overline{\psi}$ and $\overline{S}$ are the quaternion conjugates defined as

$$\overline{V} = -\mathrm{i}\frac{\phi}{v}e_0 - (V_1 e_1 + V_2 e_2 + V_3 e_3) \tag{50}$$

$$\overline{J} = -\mathrm{i}\rho v e_0 - (J_1 e_1 + J_2 e_2 + J_3 e_3) \tag{51}$$

$$\overline{\psi} = -\psi_t + \frac{\mathrm{i}}{v}(e_1\psi_1 + e_2\psi_2 + e_3\psi_3) \tag{52}$$

$$\overline{S} = -S_t + \mathrm{i}\sqrt{\frac{\epsilon}{\mu}}(e_1 S_1 + e_2 S_2 + e_3 S_3). \tag{53}$$

Hence the quaternion forms of equations (18), (19) for generalized potential, equations (23), (24) for generalized Maxwell's Dirac equation and Lorentz force equation (25) for dyons in a homogeneous medium may be expressed in terms of the following set of quaternion equations in a simple, compact and consistent manner:

$$\boxdot\,\overline{\boxdot}V = J \tag{54}$$

$$[\boxdot, G] = J \tag{55}$$

$$[\boxdot, J] = 0 \tag{56}$$

$$Q[u, G] = f \tag{57}$$

where

$$u = -\mathrm{i}u_0e_0 + u_1e_1 + u_2e_2 + u_3e_3 \tag{58}$$

$$G = -\mathrm{i}G_0e_0 + G_1e_1 + G_2e_2 + G_3e_3. \tag{59}$$

In equation (57), u, G and f are the quaternionic forms of the velocity, generalized field tensor and Lorentz force associated with dyons in a homogeneous medium. The elements of G in equation (59) are the following quaternion forms:

$$G_0 = G_{01}e_1 + G_{02}e_2 + G_{03}e_3, \tag{60}$$

$$G_1 = \mathrm{i}G_{10}e_1 + G_{12}e_2 + G_{13}e_3, \tag{61}$$

$$G_2 = \mathrm{i}G_{20}e_1 + G_{21}e_2 + G_{23}e_3, \tag{62}$$

$$G_3 = \mathrm{i}G_{30}e_1 + G_{31}e_2 + G_{32}e_2. \tag{63}$$

Equations (55), (56) and (57) may also be described as

$$[\boxdot, G_\mu] = J_\mu \tag{64}$$

and

$$Q[u, G_\mu] = f_\mu \tag{65}$$

where J_μ and f_μ are the four current and four force associated with the generalized field of dyons in a homogeneous (isotropic) medium. Let us factorize the wave operator (namely Maxwell operator) as the combination of a quaternion and its conjugate in the following manner [27]:

$$\left(-\triangle - \frac{1}{v^2}\partial_t^2\right) = \left(-\frac{\mathrm{i}}{v}\partial_t + D\right)\left(-\frac{\mathrm{i}}{v}\partial_t - D\right). \tag{66}$$

Moreover, each solution (scalar or biquaternionic) of the wave equation for generalized fields of dyons in an isotropic medium, i.e.

$$\left(\triangle + \frac{1}{v^2}\partial_t^2\right)b = 0, \tag{67}$$

can be written in a unique way as the sum of two functions A and B. In equation (67) we have described b as the generalized quantities of dyons which can be decomposed in terms of two functions A and B respectively associated with electric and magnetic charges. Equation (67) now reduces to the following set of differential equations, i.e.,

$$\left(D - \frac{\mathrm{i}}{v}\partial_t\right)A = j_e \tag{68}$$

$$\left(-D - \frac{\mathrm{i}}{v}\partial_t\right)B = j_m \tag{69}$$

associated respectively for the electric and magnetic charges of a particle described as dyons in an isotropic medium.

4. Moving dyons

Let us define the generalized charge of dyons moving in the generalized electromagnetic field as follows:

$$Q = e - \mathrm{i}\frac{g}{v} \tag{70}$$

where e is the electronic charge and g is the magnetic charge. Let us assume that a dyon is moving with a velocity $\overrightarrow{V}(t)$. Thus, the electric charge density ρ_e of a dyon may be expressed as

$$\rho_e(t, x) = e\delta(x - S(t)) \tag{71}$$

where $S(t)$ is the trajectory of the electron and the current density $\overrightarrow{j_e}$ is written as

$$\overrightarrow{j_e}(t, x) = \overrightarrow{V}(t)\rho_e(t, x). \tag{72}$$

We now use equation (66) along with the known solution of the equation, i.e.,

$$\left(\triangle + \frac{1}{v^2}\partial_t^2\right) b(t, x) = A(t)\delta(x - S(t)) \tag{73}$$

which may be expressed in terms of the following formula:

$$b(t, x) = \frac{A(\tau(t))}{4\pi|x - S(\tau(t))|(1 - M(S(t))} \tag{74}$$

where

$$M(\tau) = \frac{\langle\overrightarrow{V}(\tau), x - S(\tau)\rangle}{v|x - S(\tau)|} \tag{75}$$

is called the Mach number and τ satisfies the equation

$$\frac{|x - S(\tau)|}{v} - (t - \tau) = 0. \tag{76}$$

Similarly to the case of electron, let us describe that a monopole constituent of a dyon is also moving with a velocity $\overrightarrow{V}(t)$. Then the magnetic charge density ρ_m and magnetic current density $\overrightarrow{j_m}$ in view of duality transformations [21, 22] lead to the following expressions, i.e.,

$$\rho_m(t, x) = g\delta(x - S(t)) \tag{77}$$

and

$$\overrightarrow{j_m}(t, x) = \overrightarrow{V}(t)\frac{\rho_m(t, x)}{v^2}. \tag{78}$$

Taking into account the explicit form of ρ_e, j_e and ρ_m, j_m, equation (42) is described as follows:

$$\overline{\boxdot}\psi = \left[\frac{1}{\epsilon} + \mathrm{i}v\mu\overrightarrow{V}(t)\right]e\delta(x - S(t)) - \frac{\mathrm{i}}{\epsilon}\left[\frac{1}{v} + \mathrm{i}\frac{\overrightarrow{V}(t)}{v^2}\right]g\delta(x - S(t)). \tag{79}$$

Thus, the purely vectorial biquaternion

$$\psi(t, x) = \left(-\frac{\mathrm{i}}{v}\partial_t + D\right)b(t, x) \tag{80}$$

is a solution of equation (79) if b is a solution of equation (73) with

$$A(t) = \left[\left(\frac{e}{\epsilon} + \mathrm{i}ev\mu\overrightarrow{V}(t)\right) - \frac{\mathrm{i}}{\epsilon}\left(\frac{g}{v} + \mathrm{i}g\frac{\overrightarrow{V}(t)}{v^2}\right)\right]. \tag{81}$$

Here b and A both are complex quaternionic functions. Using equations (73)–(75) we obtain

$$b_0(t,x) = \frac{\frac{1}{\epsilon}\left(e - \mathrm{i}\frac{g}{v}\right)}{4\pi|x - S(\tau(t))|(1 - M(\tau(t))} = \frac{Q}{4\pi\epsilon|x - S(\tau(t))|(1 - M(\tau(t))} \tag{82}$$

and

$$\overrightarrow{b}(t,x) = \frac{\mathrm{i}\left(ev\mu\overrightarrow{V}(t) - \mathrm{i}\frac{g}{\epsilon}\frac{\overrightarrow{V}(t)}{v^2}\right)}{4\pi|x - S(\tau(t))|(1 - M(\tau(t))} = \frac{\mathrm{i}v\mu\overrightarrow{V}(t)Q}{4\pi|x - S(\tau(t))|(1 - M(\tau(t))}. \tag{83}$$

Thus, solution (79) reduces to a simple differentiation, i.e.

$$\psi(t,x) = \left(-\frac{\mathrm{i}}{v}\partial_t + D\right)\left[\frac{\frac{1}{\epsilon}Q + \mathrm{i}v\mu\overrightarrow{V}(t)Q}{4\pi|x - S(\tau(t))|(1 - M(\tau(t))}\right]. \tag{84}$$

We may now introduce the auxiliary functions as

$$\varphi = \frac{1}{|x - S(\tau(t))|(1 - M(\tau(t))} \tag{85}$$

$$\zeta = \frac{1}{4\pi}\left[\frac{1}{\epsilon}Q + \mathrm{i}v\mu\overrightarrow{V}(t)Q\right]. \tag{86}$$

Then

$$\psi = \left(-\frac{\mathrm{i}}{v}\partial_t + D\right)[\zeta]\cdot\varphi + \left(-\frac{\mathrm{i}}{v}\partial_t + D\right)[\varphi]\cdot\zeta. \tag{87}$$

It is easy to see that the scalar part of the expression on the right-hand side is zero. Rewriting Maxwell's equations (4) in vector form as follows:

$$\overrightarrow{\psi} = -\frac{\mathrm{i}}{v}(\partial_t\overrightarrow{\zeta}\cdot\varphi + \partial_t\varphi\cdot\overrightarrow{\zeta}) + \varphi\overrightarrow{\nabla}\times\overrightarrow{\zeta} + \zeta_0\overrightarrow{\nabla}\varphi + [\overrightarrow{\nabla}\varphi\times\overrightarrow{\zeta}]. \tag{88}$$

By the definition of $\overrightarrow{\psi}$ from equation (7), we have

$$\overrightarrow{E} = -\frac{\mathrm{i}}{v}(\partial_t\overrightarrow{\zeta}.\varphi + \partial_t\varphi.\overrightarrow{\zeta}) + \frac{1}{4\pi\epsilon}Q[\overrightarrow{\nabla}\varphi] \tag{89}$$

and

$$v\overrightarrow{B} = \varphi\overrightarrow{\nabla}\times\overrightarrow{\zeta} + [\overrightarrow{\nabla}\varphi\times\overrightarrow{\zeta}]. \tag{90}$$

To obtain the following generalized electric and magnetic field vectors $\overrightarrow{E}$ and $\overrightarrow{B}$ in explicit form we have used the following intermediate equalities, i.e.,

$$\partial_t\overrightarrow{\zeta} = \mathrm{i}\frac{\left(ev\mu\overrightarrow{V}'(\tau) - \mathrm{i}\frac{g}{\epsilon}\overrightarrow{V}'(\tau)\right)}{4\pi(1 - M(\tau))}, \tag{91}$$

$$\overrightarrow{\nabla}\times\overrightarrow{\zeta} = \frac{\mathrm{i}}{4\pi}\left[ev\mu(\overrightarrow{\nabla}\tau\times\overrightarrow{V}'(\tau)) - \mathrm{i}\frac{g}{\epsilon}(\overrightarrow{\nabla}\tau\times\overrightarrow{V}'(\tau))\right], \tag{92}$$

$$\begin{aligned}[\overrightarrow{\nabla}\varphi\times\overrightarrow{\zeta}] = \Bigg(v^2 + \frac{\mathrm{i}}{4\pi}&\left\{\frac{ev\mu[(x-s)\times\overrightarrow{V}] - \mathrm{i}\frac{g}{\epsilon}[(x-s)\times\overrightarrow{V}]}{v^2|x-s|^3(1-M)^3}\right\}\\ &\times\langle v^2 + \langle\overrightarrow{V}', x - s\rangle - |\overrightarrow{V}|^2\Bigg).\end{aligned} \tag{93}$$

As such, the solutions of the above problem for a moving dyon in terms of electric and magnetic field vectors may now be obtained in the following forms in terms of intermediate equalities, i.e.,

$$\overrightarrow{E} = \frac{\mathrm{i}}{4\pi} e\mu \left\{ \frac{\overrightarrow{V}'(\tau)}{|x-s|(1-M)^2} + \left[\frac{(\overrightarrow{V}(\tau)|x-s| - \overrightarrow{v}|x-s|)}{v|x-s|^3(1-M)^3} \right] (v^2 + \langle \overrightarrow{V}', x-s\rangle - |\overrightarrow{V}|^2) \right\}$$
$$+ \frac{g}{4\pi\epsilon v} \left\{ \frac{\overrightarrow{V}'(\tau)}{|x-s|(1-M)^2} + \left[\frac{(\overrightarrow{V}'(\tau)|x-s| - \overrightarrow{v}|x-s|)}{v|x-s|^3(1-M)^3} \right] (v^2 + \langle \overrightarrow{V}', x-s\rangle - |\overrightarrow{V}|^2) \right\} \tag{94}$$

and

$$\overrightarrow{B} = \frac{e\mu}{4\pi} \left\{ \frac{[(x-s)\times\overrightarrow{V}'(\tau)]}{v|x-s|^2(1-M)^2} + \frac{[(x-s)\times\overrightarrow{v}(\tau)]}{v^2|x-s|^3(1-M)^3} (v^2 + \langle \overrightarrow{V}', x-s\rangle - |\overrightarrow{V}|^2) \right\}$$
$$- \frac{\mathrm{i}g}{v\epsilon} \left\{ \frac{[(x-s)\times\overrightarrow{V}'(\tau)]}{v|x-s|^2(1-M)^2} + \frac{[(x-s)\times\overrightarrow{v}(\tau)]}{v^2|x-s|^3(1-M)^3} (v^2 + \langle \overrightarrow{V}', x-s\rangle - |\overrightarrow{V}|^2) \right\}. \tag{95}$$

These expressions, associated with the solutions of generalized field equations of dyons, reduce to the solutions of usual electric and magnetic field vectors in the absence of magnetic (electric) charge on dyons similar to those described by Kravchenko [27] or vice versa.

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